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A Time-Fractional Step Method for Conservation Law Related Obstacle Problems

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We are interested in approximating the solution of a first-order quasi-linear equation associated with a forced unilateral obstacle condition. With this view, we make use of the time-splitting method developed classically to compute discontinuous solutions of nonhomogeneous scalar conservation laws. Here, one proves that this fractional step method converges in L^1 to the weak entropy solution of the considered obstacle problem. In the case of the Cauchy problem, an L^1-error bound in $\mathcal{O}(\sqrt{\Delta t})$ is established. © 2001 Elsevier Science

Key Words: conservation law; obstacle problem; time-splitting method.

1. INTRODUCTION

Obstacle problems for first-order hyperbolic operators were introduced by Bensoussan and Lions [3] in 1973 as part of the study of cost-functions associated with deterministic processes. Since then, the existence and uniqueness theory, the qualitative and behavior properties for the solution of those problems have been achieved through numerous researches [2, 6, 14, 16].

In this work, we consider a general scalar conservation law associated with a forced unilateral constraint. With this view, we introduce the quasi-linear first-order operator

$$\mathbb{H}(u) \colon u \to \partial_t u + \sum_{i=1}^{p} \partial_{x_i} f_i(u) - g(u),$$

where only the dependence on u is taken into consideration in the transport and reaction terms. Then, given a real a and a measurable function u_0 such

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that $a \leq u_0$ a.e. on $\mathbb{R}^p$, $p \in \mathbb{N}^*$, we consider the initial-value problem for any real T of $]0, +\infty[$. find u satisfying the free boundary problem,

$$a \leq u \qquad \text{a.e. on } \pi_T =]0, T[\times \mathbb{R}^p, \tag{1}$$

$$\mathbb{H}(u) \geq 0 \quad \text{and} \quad (u-a)\mathbb{H}(u) = 0 \quad \text{on } \pi_T, \tag{2}$$

$$u(0, .) = u_0 \quad \text{on } \mathbb{R}^p.$$

To approximate this solution, we examine a first-order time-splitting method which consists here in alternating the exact resolution of the homogeneous equation

$$\partial_t u + \sum_{i=1}^{p} \partial_{x_i} f_i(u) = 0 \tag{3}$$

and in solving exactly the ordinary differential inequality (ODI)

$$u \geq a, \quad \partial_t u \geq g(u) \quad \text{and} \quad (u-a)(\partial_t u - g(u)) = 0. \tag{4}$$

Indeed, it is well known that the maximum principle warrants that if the initial datum for (3) satisfies (1), so it is with the associated solution. That is why constraint (1) is only considered for the resolution of the ordinary differential equation (ODE)

$$\partial_t u = g(u) \tag{5}$$

corresponding to (4).

The purpose of this article is to estimate the L^1-error bound between the exact solution to (1)–(2) and the split one given by the process (3)–(4).

2. NOTATION AND MAIN RESULT

2.1. *Assumptions on Data*

We assume the following hypotheses are fulfilled in the rest of this paper:

(h) *f_i, $i \in \{1, \ldots, p\}$, and g are respectively $\mathscr{C}^1$-class and $\mathscr{C}^0$-class functions on $\mathbb{R}$. What is more, g is Lipschitzian on $\mathbb{R}$ with a Lipschitz constant K_g and $g(0) = 0$.*

(h') *u_0 belongs to $\overline{BV}(\mathbb{R}^p) \cap L^\infty(\mathbb{R}^p)$, $a \leq u_0$ a.e. on $\mathbb{R}^p$, where $\overline{BV}(\mathbb{R}^p)$ is the space of locally integrable functions on $\mathbb{R}^p$ with finite total variation on $\mathbb{R}^p$, i.e., such that*

$$TV_{\mathbb{R}^p}(u_0) = \sup\left\{\int_{\mathbb{R}^p} u_0 \operatorname{Div} \vec{\phi}\, dx,\ \vec{\phi} \in (\mathscr{C}_c^1(\mathbb{R}^p))^p,\ \|\vec{\phi}\|_\infty \leq 1\right\} < +\infty.$$

Just note that it is the unilateral obstacle condition for the initial datum that implies that u_0 is only locally integrable on $\mathbb{R}^p$.

Let us denote by $|\cdot|_{\mathscr{E}}$ the Euclidian norm on $\mathbb{R}^p$ and by $\mathscr{B}_L$ the open ball in $\mathbb{R}^p$ centered on the origin and of radius L, and, for all t of $[0, T]$, let us write

$$M(t) = \|u_0\|_{L^\infty(\mathbb{R}^p)} \exp(K_g\, t) \quad \text{and} \quad \mathscr{N} = \sup_{-M(T)<u<M(T)} \left|\vec{f}'(u)\right|_{\mathscr{E}}.$$

Then the characteristic cone $\mathscr{K}_R$ is given by

$$\mathscr{K}_R = \{(t, x) \in \pi_T,\, x \in \mathscr{B}_{\overline{R}-\mathscr{N}t}\} \quad \text{where } \overline{R} = R + \mathscr{N}T.$$

Lastly, $\|\cdot\|_X$ stands for the L^1-norm on X and $\mathscr{C}^1_{c,+}([0, T[\times\mathbb{R}^p)$ is the space of positive $\mathscr{C}^1$-class functions with a compact support in $[0, T[\times\mathbb{R}^p$.

2.2. *Operator Splitting and Error Estimate*

Let $S(t)$, $t \in [0, T]$, be the operator that associates the exact solution $u(t, \cdot)$ of (1) and (2) to the bounded function u_0. For any strictly positive integer N, we define the time step

$$\Delta t = \frac{T}{N}.$$

That way, for all n of $\{1, \ldots, N\}$, the fractional step solution $\mathscr{F}_s(t_n)u_0$ of unilateral problem (1) and (2) at the time $t_n = n\,\Delta t$ is given by

$$\mathscr{F}_s(t_n)u_0 = (\mathscr{H}(\Delta t)\mathscr{I}(\Delta t))^n u_0, \tag{6}$$

$\mathscr{H}(t)$ and $\mathscr{I}(t)$, $t \in [0, T]$, being respectively the operators corresponding to the exact resolution of (3) and (4). The interest of such a splitting method is that, for any n of $\{1, \ldots, N\}$, $\mathscr{F}_s(t_n)u_0$ fulfills the unilateral constraint (1). Therefore, a judicious extension on each Δt-long interval permits us to construct an approximate solution satisfying the obstacle condition a.e. on π_T. Moreover, it can be compared in $L^\infty(0, T; L^1)$ to the exact solution of (1) and (2) thanks to a Kruskov-type method [9] based on a doubling of the variables and on a penalization procedure. Hence the next statement:

THEOREM 2.1. *There exists a constant C such that*, $\forall\, R > 0$,

$$\max_{1\le n\le N} \|S(t_n)u_0 - \mathscr{F}_s(t_n)u_0\|_{\mathscr{B}_{\overline{R}-\mathscr{N}t_n}} \le C\sqrt{\Delta t}\big(C_R + \max\big(\sqrt{\Delta t}, R\big)^{p-1}\big),$$

where $C_R = 1 + \|u_0\|_{\mathscr{B}_{\overline{R}}}$.

3. PRELIMINARIES TO DEMONSTRATION

3.1. *Entropy Solutions to First-Order Unilateral Problems*

We first have to define a mathematical formulation for the obstacle problem (1) and (2). Indeed, on the one hand, for a general first-order quasilinear equation it is classical to introduce the notion of an entropy solution. On the other hand, without any assumptions on the sign of the source term for $\mathbb{H}$, the introduction of an obstacle condition for the initial datum does not a priori pass on to the solution. That is why we need an entropy criterion also allowing for this constraint. To clarify the writing let us denote

$$\vec{G}(v, k) = \operatorname{sign}(v - k)\big[\vec{f}(v) - \vec{f}(k)\big],$$

$$\mathscr{L}(v, k, \phi) = |v - k|\partial_t \phi + \vec{G}(v, k) \cdot \vec{\nabla}\phi + \operatorname{sign}(v - k)g(v)\phi.$$

Thus, according to the definition given in [12] for a simple positiveness condition, we say:

DEFINITION 3.1. A bounded function u is an entropy solution for (1) and (2) relative to the bounded initial datum u_0 if and only if, (1) being satisfied,

$$\int_{\pi_T} \mathscr{L}(u, k, \phi)\, dx\, dt + \int_{\mathbb{R}^p} |u_0 - k|\phi(0, \cdot)\, dx \geq 0, \tag{7}$$

for all ϕ of $\mathscr{C}^1_{c,+}([0, T[\times\mathbb{R}^p)$ and for any real k of $[a, +\infty[$.

The uniqueness of such a solution comes from the work of Kruskov [9] and its existence, when $a = 0$, is established in [12] through the method of penalization. However, we are interested in the space and time regularities for u that are sufficient to ensure that Theorem 2.1 holds. Namely:

THEOREM 3.1. *Under hypotheses* (h) *and* (h'), $\forall\, t \in [0, T]$, $\forall\, R > 0$,

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^p)} \leq M(t), \tag{8}$$

$$\|u(t, \cdot)\|_{\mathscr{B}_{\overline{R}-\mathscr{N}t}} \leq \|u_0\|_{\mathscr{B}_{\overline{R}}} \exp(2K_g t), \tag{9}$$

$$TV_{\mathbb{R}^p}(u(t, \cdot)) \leq TV_{\mathbb{R}^p}(u_0) \exp(K_g t). \tag{10}$$

Lastly, there exists a constant C *such that*, $\forall\, h \in]0, T[$, $\forall\, t \in [0, T - h]$,

$$\|u(t + h, \cdot) - u(t, \cdot)\|_{\mathscr{B}_{\overline{R}-\mathscr{N}t}} \leq Ch\big[1 + \|u_0\|_{\mathscr{B}_{\overline{R}}}\big]. \tag{11}$$

Proof. These estimates are obtained by coming back to the viscous-penalized problem corresponding to (1) and (2) which is defined, for any strictly positive parameters ϵ and η (intended to tend to zero), by the following: find $u_{\epsilon,\eta}$ in $L^\infty(\pi_T) \cap H^2(\pi_T)$ such that

$$\mathbb{H}(u_{\epsilon,\eta}) = \epsilon\, \Delta u_{\epsilon,\eta} + \frac{1}{\eta}(u_{\epsilon,\eta} - a)^- \quad \text{a.e. on } \pi_T, \tag{12}$$
$$u_{\epsilon,\eta}(0,.) = u_{0,\epsilon} \quad \text{a.e. on } \mathbb{R}^p,$$

where $u_{0,\epsilon}$ is a standard regularization of u_0 by means of a mollifier sequence indexed on the viscous parameter; thus $u_{0,\epsilon} \geq a$ a.e. on $\mathbb{R}^p$.

Thanks to Kruskov's work [9], we know that, η being fixed, $(u_{\epsilon,\eta})_{\epsilon>0}$ is relatively compact in $\mathscr{C}([0,T]; L^1(\mathscr{B}_R))$ for any strictly positive real R. A diagonal extraction process warrants the convergence a.e. on π_T of the whole sequence $(u_{\epsilon,\eta})_{\epsilon>0}$ to the Kruskov entropy solution u_η of the first-order quasi-linear penalized problem: find u_η in $L^\infty(\pi_T) \cap L^\infty(0,T; BV(\mathbb{R}^p)) \cap \mathscr{C}([0,T]; L^1_{loc}(\mathbb{R}^p))$ such that

$$\mathbb{H}(u_\eta) = \frac{1}{\eta}(u_\eta - a)^- \quad \text{on } \pi_T, \tag{13}$$
$$u_\eta(0,\cdot) = u_0 \quad \text{on } \mathbb{R}^p.$$

First, the maximum principle or an L^1-truncation method proves (8) for $u_{\epsilon,\eta}$ (and so for u_η)—the independence with respect to η resulting from the monotonicity of the penalized operator $r \to \frac{1}{\eta}(r-a)^-$.

Now, we detail the proof of (9) for u_η with typical arguments also used to demonstrate (11): let s be in $[0,T]$ and ϕ in $\mathscr{C}^1_{c,+}([0,T] \times \mathbb{R}^p)$. One considers the $L^2(\pi_s)$-scalar product between (12) and the test-function $\mathrm{sign}_\lambda(u_{\epsilon,\eta})\phi$, where $\mathrm{sign}_\lambda(\cdot)$ is the approximation of the function sign defined by, $\forall\, x \geq 0$,

$$\mathrm{sign}_\lambda(x) = \min\left(\frac{x}{\lambda}, 1\right), \qquad \mathrm{sign}_\lambda(-x) = -\mathrm{sign}_\lambda(x).$$

When λ goes to 0^+, the Lebesgue dominated convergence theorem, the Saks lemma (to study the transport term), and the fact that $g(0) = 0$ all ensure that, $\forall\, s \in [0,T]$,

$$\begin{aligned}
&\int_{\mathbb{R}^p} |u_{\epsilon,\eta}(s,x)|\phi(s,x)\,dx - \int_{\mathbb{R}^p} |u_{0,\epsilon}(x)|\phi(0,x)\,dx \\
&\quad \leq \int_{\pi_s} \frac{1}{\eta}(u_{\epsilon,\eta} - a)^- \phi\,dx\,dt + K_g \int_{\pi_s} |u_{\epsilon,\eta}|\phi\,dx\,dt \\
&\qquad + \epsilon \int_{\pi_s} |u_{\epsilon,\eta}|\Delta\phi\,dx\,dt + \int_{\pi_s} |u_{\epsilon,\eta}|\{\partial_t\phi + \mathscr{N}|\vec{\nabla}\phi|_{\mathscr{E}}\}\,dx\,dt.
\end{aligned}$$

In this inequality, it is possible to pass to the limit with respect to ϵ. Then, referring to the idea of Kruskov, we consider a regular approximation χ_δ of the characteristic function of $\mathscr{K}_R$ such that

$$\chi_\delta \equiv 0 \quad \text{outside } \mathscr{K}_R \qquad \text{and} \qquad \partial_t\chi_\delta + \mathscr{N}|\vec{\nabla}\chi_\delta|_{\mathscr{E}} \equiv 0 \quad \text{in } \mathscr{K}_R, \tag{14}$$

and we choose ϕ equal to χ_δ. Thence the limit with respect to δ gives

$$\int_{\mathscr{B}_{\overline{R}-\mathcal{N}s}} |u_\eta(s,x)|\,dx - \int_{\mathscr{B}_{\overline{R}}} |u_0(x)|\,dx$$
$$\leq \int_0^s \int_{\mathscr{B}_{\overline{R}-\mathcal{N}t}} \frac{1}{\eta}(u_\eta - a)^-\,dx\,dt + K_g \int_0^s \int_{\mathscr{B}_{\overline{R}-\mathcal{N}t}} |u_\eta|\,dx\,dt.$$

Thus, to establish (9) for u_η, we need an L^1-majoration of the penalized term in (13). It is obtained by taking the $L^2(\pi_s)$-scalar product between (12) and the test-function $-[\mathrm{sign}_\lambda(u_{\epsilon,\eta} - a)]^-\chi_\delta$. That way, the same techniques as before permit us to state that, for all s in $[0, T]$,

$$\int_0^s \int_{\mathscr{B}_{\overline{R}-\mathcal{N}t}} \frac{1}{\eta}(u_\eta - a)^-\,dx\,dt \leq K_g \int_0^s \int_{\mathscr{B}_{\overline{R}-\mathcal{N}t}} |u_\eta|\,dx\,dt,$$

and (9) for u_η follows from Gronwall's lemma.

The proof of (11) for u_η is based on the same principles and can be resumed in the next two steps:

First, given h in $]0, T[$, we take the $L^2(\pi_h)$-scalar product between (12) and the test-function $\mathrm{sign}_\lambda(u_{\epsilon,\eta} - u_{0,\epsilon})\chi_\delta$. It provides an L^1-estimate of the difference $u_{\epsilon,\eta}(h,.) - u_{0,\epsilon}$, independently of ϵ and η.

Second, we consider (12) at the neighbouring times t and $t+h$, $t \in [0, T-h]$. The two resulting equalities are subtracted. Then, s being an element of $[0, T-h]$, we take the $L^2(\pi_s)$-scalar product with the function $\mathrm{sign}_\lambda(u_{\epsilon,\eta}(t+h,\cdot) - u_{\epsilon,\eta}(t,\cdot))\chi_\delta$.

Eventually, we demonstrate (10) for u_η through the proof given in [7]—by differentiating (12) with respect to x_i—and thanks to the lower semicontinuity of the total variation for the L^1-norm; the independence with respect to η results from the monotonicity of the penalized operator.

Now, since u_η fulfills estimates (8)–(11) independently of η, a diagonal extraction process is used again to demonstrate the existence of a function u of $L^\infty(\pi_T) \cap L^\infty(0, T; BV(\mathbb{R}^p)) \cap \mathscr{C}([0, T]; L^1_{loc}(\mathbb{R}^p))$ such that $u \geq a$ a.e. on π_T, thanks to the estimate of the penalized term in (13). Entropy formulation (7) for u is proved by taking the $L^2(\pi_T)$-scalar product between (12) and the test-function $\mathrm{sign}_\lambda(u_{\epsilon,\eta} - k)\phi$, where ϕ belongs to $\mathscr{C}^1_{c,+}([0, T[\times\mathbb{R}^p)$, k is a real element of $[a, +\infty[$. Noting that $-\mathrm{sign}_\lambda(u_{\epsilon,\eta} - k)(u_{\epsilon,\eta} - a)^-\phi$ is nonnegative a.e. on π_T, it is possible to pass to the limit respectively when ϵ, η, and λ goes to 0^+. To conclude, we remember that the uniqueness theorem (which follows from Kruskov's work) proves that the whole sequence $(u_\eta)_{\eta>0}$ converges to u a.e. on π_T, in $L^1(\mathscr{B}_R)$ and in $\mathscr{C}([0, T]; L^1(\mathscr{B}_R))$, $\forall R > 0$. That way u fulfills (8)–(11). ∎

In the demonstration of entropy condition (7) for u, we can also consider the $L^2(\pi_T)$-scalar product between (12) and $\mathrm{sign}_\lambda(u_{\epsilon,\eta} - k)\phi\chi_\delta$. Then, by passing to the limit with respect to ϵ, η, λ, and δ—through specific properties (14) for χ_δ—we obtain the next relation, which will be useful in the rest of the paper:

COROLLARY 3.1. *For all t_1 and t_2 of $[0, T]$, $t_1 < t_2$, for any real k in $[a, +\infty[$ and for any function ϕ of $\mathscr{C}^1_{c,+}([0,T] \times \mathbb{R})$,*

$$\int_{\mathscr{K}_R^{t_1,t_2}} \mathscr{L}(u,k,\phi)\,dx\,dt - \left[\int_{\mathscr{B}_{\overline{R}-\mathscr{N}t}} |u-k|\phi\,dx\right]_{t_1}^{t_2} \geq 0, \tag{15}$$

where $[w]_{t_1}^{t_2} = w(t_2) - w(t_1)$ and $\mathscr{K}_R^{t_1,t_2}$ is the truncation of $\mathscr{K}_R$ between t_1 and t_2.

3.2. *Weak Solutions to an ODI*

Now we have to study the operator $\mathscr{I}(t)$ given in (6), that is, to specify some existence, uniqueness, and behavior properties for the solution of ODI (4). For this purpose, let us write

$$K_a = \{w \in L^\infty(\mathbb{R}^p), w \geq a \text{ a.e. on } \mathbb{R}^p\}.$$

Hence we have the next statement:

THEOREM 3.2. *Let w_0 be an element of K_a. Obstacle problem (4) associated with the initial datum w_0 has a unique bounded solution w, which belongs to $H^1(0,T;L^\infty(\mathbb{R}^p))$ and which is characterized by the variational formulation*

$$\forall\, t \in [0,T], \qquad w(t,\cdot) \in K_a, \tag{16}$$

$$\text{for a.e. } t \in]0,T[, \forall\, v \in K_a, \quad \partial_t w(v-w) \geq g(w)(v-w) \quad \text{a.e. on } \mathbb{R}^p, \tag{17}$$

$$w(0,\cdot) = w_0 \quad \text{a.e. on } \mathbb{R}^p.$$

Besides, if $\hat{u}$ is the solution of ordinary differential equation (5) *related to the initial datum w_0, then*

$$w = \max(\hat{u}, a).$$

Proof. *Uniqueness*: Let w_1 and w_2 be elements of $H^1(0,T;L^\infty(\mathbb{R}^p))$ such that (16)–(17) hold and are associated with the same initial datum w_0. In inequality (17) satisfied by w_1 (resp. w_2) one chooses the test-function w_2 (resp. w_1). By adding up the resulting relations and integrating over $[0,t]$, for any real t in $[0,T]$, the uniqueness property follows from the Gronwall lemma since g is Lipschitzian.

Existence: Let $\hat{u}$ be the solution of ODE (5) related to the initial datum w_0 and let $w \equiv \max(\hat{u}, a)$. Clearly w fulfills the obstacle condition (16).

Moreover:

• If $g(a) \geq 0$ then $g(a)(\hat{u} - a)^- \geq 0$. Thus, by multiplying (5) with the function $-[\text{sign}_\lambda(\hat{u} - a)]^-$ and by integrating from 0 to s, $s \in [0, T]$, the Gronwall lemma allows us to conclude that $\hat{u} \geq a$ a.e. on π_T. Thence $w = \hat{u}$ and w fulfills (17).

• If $g(a) \leq 0$ then, for all v of K_a, $g(a)(v - a) \leq 0$ a.e. on $\mathbb{R}^p$. Thus w satisfies (17). ∎

These results prove that the existence and regularity properties for the solution of an ODE provide those of the corresponding unilateral obstacle problem by the mean of a truncation operation. Thence, taking into account the definition and the Lipschitzian condition for the function $u \to \max(u, a)$, the next property holds:

PROPOSITION 3.1. *Let w_0 be in $\overline{BV}(\mathbb{R}^p) \cap L^\infty(\mathbb{R}^p)$, $w_0 \geq a$ a.e. on $\mathbb{R}^p$. The solution w of* (4) *associated with initial datum w_0 belongs to $L^\infty(\pi_T) \cap L^\infty(0, T; BV(\mathbb{R}^p)) \cap \mathscr{C}([0, T]; L^1_{loc}(\mathbb{R}^p))$. Furthermore, $\forall\, t \in [0, T]$, $\forall\, R > 0$,*

$$\|w(t, .)\|_{L^\infty(\mathbb{R}^p)} \leq \|w_0\|_{L^\infty(\mathbb{R}^p)} \exp(K_g t).$$

$$\|w\|_{\mathscr{B}_R} \leq \|w_0\|_{\mathscr{B}_R} \exp(K_g t),$$

$$TV_{\mathbb{R}^p}(w(t, .)) \leq TV_{\mathbb{R}^p}(w_0) \exp(K_g t),$$

and there exists a constant C such that $\forall\, h \in]0, T[$, $\forall\, t \in [0, T - h]$,

$$\|w(t + h, .) - w(t, .)\|_{\mathscr{B}_R} \leq C\, h\, \|w_0\|_{\mathscr{B}_R}.$$

4. STUDY OF THE TIME-FRACTIONAL STEP METHOD

4.1. *Definition and Properties of a Time-Step Function*

First-order fractional step methods for nonhomogeneous scalar conservation laws were first introduced by Godounov [8] and Strang [18]. Since then, Crandall and Majda [5] have studied the convergence toward the weak entropy solution and Tang and Teng [20] have investigated an L^1-convergence rate in $\mathscr{O}(\sqrt{\Delta t})$. Langseth *et al.* [11] have established an L^1-error bound in $\mathscr{O}(\Delta t)$. That result has been extended by Tang [19] in the case of a stiff source term and by Peyroutet [17] and Madaune-Tort and Peyroutet [15] for the Dirichlet problem.

In previous papers [11, 17, 15, 19], the L^1-error bound results from a Kruskov-type estimate, on each Δt-long interval, between the entropy solution of the nonhomogeneous problem and the translated solution—by means of the source function—of the homogeneous corresponding equation. It leads one to consider two entropy formulations associated with two different flux functions and reaction terms. The $\mathcal{O}(\Delta t)$-error bound obtained this way can be explained through the general framework of Kruskov's estimates for scalar conservation laws given by Bouchut and Perthame [4].

Here, when changing the flux function and introducing a reaction term in (3), we cannot not be sure that the approximate solution obtained through a translation process fulfills unilateral constraint (1) on each Δt-long interval. Thus, the latter method cannot be applied to the present situation and we come back to the demonstration of Tang and H.Teng [20] to justify an L^1-convergence rate in $\mathcal{O}(\sqrt{\Delta t})$ for the splitting solution (6). With this view, we refer to the notation introduced by Crandall and Majda [5] to define the time-continuous step function $u_\Delta(t,\cdot)$ on $[0,T]$ by

$$u_\Delta(t,\cdot) = \begin{cases} \mathcal{I}(2(t-t_n))\mathcal{F}_s(t_n)u_0 & \text{if} \quad t \in [t_n, t_{n+\frac{1}{2}}[, \\ \mathcal{H}(2(t-t_{n+\frac{1}{2}}))\mathcal{I}(\Delta t)\mathcal{F}_s(t_n)u_0 & \text{if} \quad t \in [t_{n+\frac{1}{2}}, t_{n+1}[, \end{cases} \tag{18}$$

where $t_{n+1/2} = (n+\frac{1}{2})\Delta t$.

That way, estimates collected in Theorem 3.1 and in Proposition 3.1 ensure that, for any n of $\{1,\ldots,N\}$,

$$\|u_\Delta(t_n,\cdot)\|_{L^\infty(\mathbb{R}^p)} \le C_n\|u_0\|_{L^\infty(\mathbb{R}^p)}, \quad TV_{\mathbb{R}^p}(u_\Delta(t_n,\cdot)) \le C_n TV_{\mathbb{R}^p}(u_0),$$

and

$$\|u_\Delta(t_n,\cdot)\|_{\mathcal{B}_{\overline{R}-\mathcal{N}t_n}} \le C_n\|u_0\|_{\mathcal{B}_{\overline{R}}},$$

with $C_n = \exp(K_g n\Delta t)$.

Therefore it is clear that u_Δ belongs to $L^\infty(\pi_T)\cap L^\infty(0,T;BV(\mathbb{R}^p))\cap \mathcal{C}([0,T];L^1_{loc}(\mathbb{R}^p))$. Besides, the next property holds:

PROPOSITION 4.1. $\forall\, t\in[0,T]$, $\forall\, R>0$,

$$\|u_\Delta(t,\cdot)\|_{L^\infty(\mathbb{R}^p)} \le M(T), \tag{19}$$

$$\|u_\Delta(t,\cdot)\|_{\mathcal{B}_{\overline{R}-\mathcal{N}t}} \le \|u_0\|_{\mathcal{B}_{\overline{R}}}\exp(K_gT), \tag{20}$$

$$TV_{\mathbb{R}^p}(u_\Delta(t,\cdot)) \le TV_{\mathbb{R}^p}(u_0)\exp(K_gT). \tag{21}$$

Moreover, $\forall\, h\in]0,T[$, $\forall\, t\in[0,T-h]$,

$$\|u_\Delta(t+h,\cdot)-u_\Delta(t,\cdot)\|_{\mathcal{B}_{\overline{R}-\mathcal{N}t}} \le C\,h[1+\|u_0\|_{\mathcal{B}_{\overline{R}}}], \tag{22}$$

where C is a constant independent of any parameter.

4.2. *Demonstration of the Error Estimate*

We are now able to establish the main result of Theorem 2.1. For that purpose, we define a mollifier sequence $(\rho_{q,\mu})_{\mu>0}$, for any q of $\mathbb{N}^*$, such that ρ_q is a function of $\mathcal{C}^\infty_{c,+}(\mathbb{R}^q)$ and

$$\mathrm{Supp}(\rho_q) \subset \{x\in\mathbb{R}^q, |x|_{\mathcal{E}} \le 1\}, \quad \rho_q(-x) = \rho_q(x),$$

and

$$\int_{\mathbb{R}^q} \rho_q(x)dx = 1.$$

Then, for each strictly positive real μ, $\rho_{q,\mu}(y) = \frac{1}{\mu^q}\rho_q(\frac{y}{\mu})$.
In addition, we use the notation introduced in [20] and we set

$$\begin{aligned}
\omega_\mu(x, x', t, t') &= \rho_{1,\mu}(t-t')\rho_{p,\mu}(x-x') \quad \text{for any } (t, x) \text{ and } (t', x') \text{ of } \pi_T,\\
\Lambda^*(w, v) &= -\int_{\mathscr{K}_R^n} \{|w - v'|\partial_t \omega_\mu + \vec{G}(w, v') \cdot \vec{\nabla}_x \omega_\mu\}\, d\pi\\
&\quad + \Big[\int_{\mathscr{B}_{\overline{R}-\mathscr{N}t}} |w - v'|\omega_\mu\, dx\Big]_0^{t_n},\\
\Lambda(w, v) &= \int_{\mathscr{K}_R^n} \Lambda^*(w, v)\, d\pi'.
\end{aligned}$$

where w and v' denote respectively $w(t, x)$ and $v(t', x')$, and $\mathscr{K}_R^n$ is the truncation of the characteristic cone $\mathscr{K}_R$ at the time t_n. Moreover $d\pi = dx\,dt$, $d\pi' = dx'\,dt'$, and C designates **any constant independent of parameters** n, μ, R, **and** Δt.

The demonstration—based on that developed in [20]—comes from a Kuznetsov-type inequality [10] which leads us to look for a majoration of $\Lambda(u_\Delta, u)$ and $\Lambda(u, u_\Delta)$ and a minoration for $\Lambda(u_\Delta, u) + \Lambda(u, u_\Delta)$. Thence we have the following three steps:

First Step. We begin with the essential result:

LEMMA 4.1. $\forall\, m \in \{0, \ldots, n-1\}$, $\forall\, k \in [a, +\infty[$, $\forall\, \phi \in \mathscr{C}^1_{c,+}([0, T] \times \mathbb{R}^p)$,

$$\begin{aligned}
&\int_{t_m}^{t_{m+1/2}} \int_{\mathbb{R}^p} \{|u_\Delta - k|\partial_t \phi + 2\,\mathrm{sign}(u_\Delta - k)g(u_\Delta)\phi\}\, d\pi\\
&\qquad \geq \int_{\mathbb{R}^p} [|u_\Delta - k|\phi]_{t_m}^{t_{m+1/2}}\, dx, \qquad (23)\\
&\int_{t_{m+1/2}}^{t_{m+1}} \int_{\mathbb{R}^p} \big\{|u_\Delta - k|\partial_t \phi + 2\vec{G}(u_\Delta, k) \cdot \vec{\nabla}\phi\big\}\, d\pi\\
&\qquad \geq \int_{\mathbb{R}^p} [|u_\Delta - k|\phi]_{t_{m+1/2}}^{t_{m+1}}\, dx. \qquad (24)
\end{aligned}$$

proof. Let us consider, for any real r in $[0, \Delta t]$

$$u_{\mathscr{J}}^{(m)}(r, x) = \mathscr{J}(r)\mathscr{F}_s(t_m)u_0 \qquad \text{and} \qquad u_{\mathscr{H}}^{(m)}(r, x) = \mathscr{H}(r)\mathscr{J}(\Delta t)\mathscr{F}_s(t_m)u_0.$$

Using the change of variable $r = 2(t - t_m)$ and denoting $\psi^{(m)}(r, x) = \phi(t_m + \frac{r}{2}, x)$ provides a.e. on $\mathbb{R}^p$

$$\int_{t_m}^{t_{m+1/2}} \{|u_\Delta - k|\partial_t\phi + 2\,\mathrm{sign}(u_\Delta - k)g(u_\Delta)\phi\}\,dt$$
$$= \int_0^{\Delta t} \{|u_{\mathcal{J}}^{(m)} - k|\partial_r\psi^{(m)} + \,\mathrm{sign}(u_{\mathcal{J}}^{(m)} - k)g(u_{\mathcal{J}}^{(m)})\psi^{(m)}\}\,dr.$$

An integration by parts allows us to turn the right-side member into

$$\big[|u_{\mathcal{J}}^{(m)} - k|\psi^{(m)}\big]_0^{\Delta t} - \int_0^{\Delta t} \mathrm{sign}(u_{\mathcal{J}}^{(m)} - k)\big(\partial_t u_{\mathcal{J}}^{(m)} - g(u_{\mathcal{J}}^{(m)})\big)\psi^{(m)}\,dr.$$

To study the sign of the integrated term we refer to the properties of Theorem 3.2:

- If $g(a) \geq 0$, then $u_{\mathcal{J}}^{(m)} = \hat{u}^{(m)}$, where $\hat{u}^{(m)}$ is the solution of ODE (5) corresponding to the initial datum $\mathcal{F}_s(t_m)u_0$. So

$$\int_0^{\Delta t} \mathrm{sign}(u_{\mathcal{J}}^{(m)} - k)\big(\partial_t u_{\mathcal{J}}^{(m)} - g(u_{\mathcal{J}}^{(m)})\big)\psi^{(m)}\,dr = 0.$$

- If $g(a) \leq 0$, then a.e. on $]0, \Delta t[\times\mathbb{R}^p$,

$$\mathrm{sign}(u_{\mathcal{J}}^{(m)} - k)\big(\partial_t u_{\mathcal{J}}^{(m)} - g(u_{\mathcal{J}}^{(m)})\big) = -\,\mathrm{sign}(a - k)g(a)\,\mathrm{sign}^-(\hat{u}^{(m)} - a).$$

Anyway, since k belongs to $[a, +\infty[$, we conclude that

$$\int_0^{\Delta t} \mathrm{sign}(u_{\mathcal{J}}^{(m)} - k)\big(\partial_t u_{\mathcal{J}}^{(m)} - g(u_{\mathcal{J}}^{(m)})\big)\psi^{(m)}\,dr \leq 0.$$

Inequality (23) follows.

Moreover, the change of variable $r = 2(t - t_{m+1/2})$, the definition of u_Δ, and the fact that $u_{\mathcal{H}}^{(m)}$ is the unique entropy solution of homogeneous equation (3) corresponding to the initial datum $\mathcal{J}(\Delta t)(\mathcal{H}(\Delta t)\mathcal{J}(\Delta t))^m u_0$ are used to establish (24), which completes the proof of Lemma 3.1. ∎

Now let us refer to χ_δ, the regular approximation of the characteristic cone $\mathcal{K}_R^n$. In relations (23) and (24) we choose $\phi = \omega_\mu\chi_\delta$ and $k = u'$ (which is possible since $u' \geq a$ a.e. on π_T). As (14) holds we may pass to the limit with respect to δ. Hence, in (23) and (24) the integration field is respectively turned into $\mathcal{K}_R^{m,\,m+1/2}$ and $\mathcal{K}_R^{m+1/2,\,m+1}$, where $\mathcal{K}_R^{a,\,b}$ $(a < b)$ is the truncation of $\mathcal{K}_R^n$ between a and b. Then, from integrating over $\mathcal{K}_R^n$

with respect to variables (t', x'), summing on m from 0 to $n-1$, and adding (23)–(24) we infer the inequality

$$\Lambda(u_\Delta, u) \leq 2\sum_{m=0}^{n-1}\int_{\mathscr{K}_R^n}\bigg\{\int_{\mathscr{K}_R^{m,\,m+1/2}} \operatorname{sign}(u_\Delta - u')g(u_\Delta)\omega_\mu\, d\pi + \bigg(\int_{\mathscr{K}_R^{m+1/2,\,m+1}} \vec{G}(u_\Delta, u')\cdot\vec{\nabla}_x\omega_\mu\, d\pi - \int_{\mathscr{K}_R^{m,\,m+1/2}} \vec{G}(u_\Delta, u')\cdot\vec{\nabla}_x\omega_\mu\, d\pi\bigg)\bigg\}\, d\pi'.$$

The first right-hand-side term is transformed by exchanging notations (t, x) and (t', x'). The second one, denoted I, is bounded by swapping variables t and t' respectively in $\tau = t - \frac{\Delta t}{2}$ and $\tau' = t' - \frac{\Delta t}{2}$ and by noting that one has $\vec{\nabla}_x\omega_\mu(x, x', \tau, \tau') = \vec{\nabla}_x\omega_\mu(x, x', t, t')$, it follows that

$$I = I_1 + I_2 + I_3$$

with

$$I_1 = \sum_{m=0}^{n-1}\int_{\mathscr{K}_R^{m+1/2,\,m+1}}\int_{\mathscr{K}_R^{1/2}} \vec{G}(u_\Delta, u')\cdot\vec{\nabla}_x\omega_\mu\, d\pi'\, d\pi,$$

$$I_2 = -\sum_{m=0}^{n-1}\int_{\mathscr{K}_R^{m+1/2,\,m+1}}\int_{\mathscr{K}_R^{1/2,\,n}}\{\vec{G}(u_\Delta(\tau,\cdot), u(\tau',\cdot)) - \vec{G}(u_\Delta, u')\}\cdot\vec{\nabla}_x\omega_\mu d\pi'\, d\pi,$$

$$I_3 = -\sum_{m=0}^{n-1}\int_{\mathscr{K}_R^{m+1/2,\,m+1}}\int_{\mathscr{K}_R^{n,\,n+1/2}} \vec{G}(u_\Delta(\tau, x), u(\tau', x'))\cdot\vec{\nabla}_x\omega_\mu\, d\pi'\, d\pi.$$

Due to the Lipschitzian property of $\vec{G}$,

$$|I_1| \leq C\int_{\mathscr{K}_R^n}\int_{\mathscr{K}_R^{1/2}}(|u_\Delta| + |u'|)\rho_{1,\,\mu}(t - t')\big|\vec{\nabla}_x\rho_{p,\,\mu}(x - x')\big|_{\mathscr{E}}\, d\pi'\, d\pi.$$

Accordingly, by taking into account the fact that $\int_{\mathbb{R}^p}|\vec{\nabla}_x\rho_{p,\,\mu}(x - x')|_{\mathscr{E}}\, dx$ is bounded by $\frac{C}{\mu}$ and (9) and (20) hold for u and u_Δ, one gets

$$|I_1| \leq C_R\frac{C}{\mu}\int_0^{t_n}\int_0^{\Delta t/2}\rho_{1,\,\mu}(t - t')\, dt'\, dt \leq C_R C\frac{\Delta t}{\mu},$$

with $C_R = 1 + \|u_0\|_{\mathscr{B}_{\bar{R}}}$.

Similarly I_3 has the same bound, and to estimate I_2 we remark that the Lipschitz condition for $\vec{G}$ gives

$$|G_i(a, b) - G_i(c, d)| \leq M_{f_i'}(|a - c| + |b - d|),$$

where $M_{f_i'} = \sup_{|\tau|\leq M(T)} |f_i'(\tau)|$, $i \in \{1, \ldots, p\}$. Thence,

$$|I_2| \leq \frac{C}{\mu} \int_{\Delta t/2}^{t_n} \int_{\Delta t/2}^{t_n} (\|u_\Delta(\tau, \cdot) - u_\Delta(t, \cdot)\|_{\mathscr{B}_{\overline{R}-\mathscr{N}t}} + \|u(\tau', \cdot) - u(t', \cdot)\|_{\mathscr{B}_{\overline{R}-\mathscr{N}t}})\, dt'\, dt.$$

Now (11) and (22) provide $|I_2| \leq C_R C \frac{\Delta t}{\mu}$.

By adding up I_1, I_2, and I_3, one obtains the existence of a constant C such that

$$\Lambda(u_\Delta, u) \leq 2 \sum_{m=0}^{n-1} \int_{\mathscr{K}_R^{m,\, m+1/2}} \int_{\mathscr{K}_R^{n}} \operatorname{sign}(u_\Delta' - u) \times g(u_\Delta')\, \omega_\mu \, d\pi\, d\pi' + C_R C \frac{\Delta t}{\mu}, \tag{25}$$

which completes our first step.

Second Step. We seek a majoration of $\Lambda(u_\Delta, u)$. To do that, let us state:

LEMMA 4.2. *There exists a constant C such that*

$$\Lambda(u, u_\Delta) - 2 \sum_{m=0}^{n-1} \int_{\mathscr{K}_R^{m,\, m+1/2}} \Lambda^*(u, u_\Delta)\; d\pi' \leq C_R C \left(\Delta t + \frac{\Delta t}{\mu} \right).$$

Proof. In this inequality, the definition of Λ and Λ^* allows us to express the left-hand-side term through the difference

$$\sum_{m=0}^{n-1} \left(\int_{\mathscr{K}_R^{m+1/2,\, m+1}} \Lambda^*(u, u_\Delta)\, d\pi' - \int_{\mathscr{K}_R^{m,\, m+1/2}} \Lambda^*(u, u_\Delta)\, d\pi' \right),$$

which is bounded in the same spirit as in the first step. Indeed, the reasoning—based on L^1_{loc}-estimates (9)–(20) and on time-continuity properties in L^1_{loc} (11)–(22) respectively for u and u_Δ—ensures the existence of a constant C such that

$$\sum_{m=0}^{n-1} \int_{\mathscr{K}_R^{n}} \left(\int_{\mathscr{K}_R^{m+1/2,\, m+1}} |u - u_\Delta'| \partial_t \omega_\mu \, d\pi - \int_{\mathscr{K}_R^{m,\, m+1/2}} |u - u_\Delta'| \partial_t \omega_\mu \, d\pi \right) d\pi' \leq C_R C \frac{\Delta t}{\mu},$$

and it is the same for the term with $\vec{\nabla}_x \omega_\mu$. Moreover,

$$\sum_{m=0}^{n-1} \int_{\mathscr{K}_R^{m+1/2,\, m+1}} \left[\int_{\mathscr{B}_{\overline{R}-\mathscr{N}t}} |u - u_\Delta'| \omega_\mu \, dx \right]_0^{t_n} d\pi' - \int_{\mathscr{K}_R^{m,\, m+1/2}} \left[\int_{\mathscr{B}_{\overline{R}-\mathscr{N}t}} |u - u_\Delta'| \omega_\mu \, dx \right]_0^{t_n} d\pi' \leq C_R C\, \Delta t.$$

∎

Now, since by construction u_Δ fulfills unilateral constraint (1), we may choose in relation (15) for u, $k = u'_\Delta$, $t_1 = 0$, $t_2 = t_n$, and $\phi = \omega_\mu$. Hence,

$$\Lambda^*(u, u_\Delta) \le \int_{\mathscr{K}_R^n} \operatorname{sign}(u - u'_\Delta) g(u) \omega_\mu \, d\pi.$$

We integrate this inequality over $\mathscr{K}_R^{m,\, m+1/2}$ with respect to the variables (t', x') and we sum from 0 to $n-1$. Lemma 4.2 thus leads to

$$\begin{aligned}\Lambda(u, u_\Delta) \le 2 \sum_{m=0}^{n-1} \int_{\mathscr{K}_R^{m,\, m+1/2}} \int_{\mathscr{K}_R^n} \operatorname{sign}(u - u'_\Delta) g(u) \omega_\mu \, d\pi \, d\pi' \\ + C_R C\left(\Delta t + \frac{\Delta t}{\mu}\right),\end{aligned} \tag{26}$$

which completes our second step.

Third Step. A minoration of the sum $\Lambda(u_\Delta, u) + \Lambda(u, u_\Delta)$ is provided by the next Kuznetsov-type lemma:

LEMMA 4.3. $\forall \mu > 0, \forall t_n \in [\mu, T], \forall R > 0,$

$$\begin{aligned}\|u(t_n, \cdot) - u_\Delta(t_n, \cdot)\|_{\mathscr{B}_{\overline{R} - \mathscr{N} t_n}} \le \Lambda(u_\Delta, u) + \Lambda(u, u_\Delta) \\ + C\mu(C_R + \max(\mu, R)^{p-1}).\end{aligned}$$

Proof. In $\Lambda(u, u_\Delta)$ we swap variables (t, x) and (t', x'). By adding with $\Lambda(u_\Delta, u)$, the properties of ω_μ lead to

$$\Lambda(u_\Delta, u) + \Lambda(u, u_\Delta) = \int_{\mathscr{K}_R^n} \left[\int_{\mathscr{B}_{\overline{R} - \mathscr{N} t}} (|u - u'_\Delta| + |u_\Delta - u'|) \omega_\mu \, dx \right]_0^{t_n} d\pi'.$$

Let us focus on the term corresponding to the time t_n (the study for $t = 0$ being similar). We introduce the decomposition $I_1 + I_2$, where

$$I_1 = \int_{\mathscr{K}_{R+\mu}^n} \int_{\mathscr{B}_{\overline{R} - \mathscr{N} t_n}} (|u(t_n, x) - u'_\Delta| + |u_\Delta(t_n, x) - u'|) \omega_\mu(t_n, \cdot) \, dx \, d\pi',$$

and

$$I_2 = -\int_{\mathscr{K}_{R+\mu}^n \setminus \mathscr{K}_R^n} \int_{\mathscr{B}_{\overline{R} - \mathscr{N} t_n}} (|u(t_n, x) - u'_\Delta| + |u_\Delta(t_n, x) - u'|) \omega_\mu(t_n, \cdot) \, dx \, d\pi'.$$

Therefore, thanks to (8) and (19),

$$|I_2| \le C \int_{\mathscr{K}_{R+\mu}^n \setminus \mathscr{K}_R^n} \int_{\mathscr{B}_{\overline{R} - \mathscr{N} t_n}} \omega_\mu(t_n, t', x, x') \, dx \, d\pi' \le C \ \operatorname{meas}\big(\mathscr{B}_{R+\mu}^n \setminus \mathscr{B}_R^n\big).$$

Moreover,

$$\operatorname{meas}\big(\mathscr{B}_{R+\mu}^n \setminus \mathscr{B}_R^n\big) \le C\mu \sum_{k=0}^{p-1} (R+\mu)^k R^{p-1-k} \le Cp\mu \max(\mu, R)^{p-1}.$$

Besides, since $\int_{\mathscr{K}^n_{R+\mu}} \omega_\mu(t_n, t', x, x')\, d\pi' = \frac{1}{2}$, the integral I_1 is rewritten

$$\begin{aligned} I_1 = &\int_{\mathscr{B}_{\overline{R}-\mathscr{N}t_n}} |u(t_n, \cdot) - u_\Delta(t_n, \cdot)|\, dx \\ &+ \int_{\mathscr{K}^n_{R+\mu}} \int_{\mathscr{B}_{\overline{R}-\mathscr{N}t_n}} (|u(t_n, x) - u'_\Delta| - |u(t_n, x) - u_\Delta(t_n, x)|)\omega_\mu(t_n, \cdot)\, dx\, d\pi' \\ &+ \int_{\mathscr{K}^n_{R+\mu}} \int_{\mathscr{B}_{\overline{R}-\mathscr{N}t_n}} (|u_\Delta(t_n, x) - u'| - |u(t_n, x) - u_\Delta(t_n, x)|)\omega_\mu(t_n, \cdot)\, dx\, d\pi'. \end{aligned}$$

Accordingly,

$$\begin{aligned} I_1 \geq &\int_{\mathscr{B}_{\overline{R}-\mathscr{N}t_n}} |u(t_n, x) - u_\Delta(t_n, x)|\, dx \\ &- \int_{\mathscr{K}^n_{R+\mu}} \int_{\mathscr{B}_{\overline{R}-\mathscr{N}t_n}} (|u_\Delta(t_n, x) - u'_\Delta| + |u(t_n, x) - u'|)\omega_\mu(t_n, \cdot)\, dx\, d\pi'. \end{aligned}$$

The integration on $\mathscr{K}^n_{R+\mu}$ is split into an integration on $\mathscr{K}^n_R$ and on $\mathscr{K}^n_{R+\mu} \backslash \mathscr{K}^n_R$, the latter being bounded by referring to the study of I_2.

Finally, due to the term corresponding to the time $t = 0$, we get

$$\begin{aligned} \int_{\mathscr{B}_{\overline{R}-\mathscr{N}t_n}} |u(t_n, x) - u_\Delta(t_n, x)|\, dx \leq\ & C\mu \max(\mu, R)^{p-1} + \Lambda(u_\Delta, u) + \Lambda(u, u_\Delta) \\ &+ \int_{\mathscr{K}^n_R} \int_{\mathscr{B}_{\overline{R}-\mathscr{N}t_n}} \big(|u_\Delta(t_n, x) - u'_\Delta| + |u(t_n, x) - u'|\big)\omega_\mu(t_n, t', x, x')\, dx\, d\pi' \\ &+ \int_{\mathscr{K}^n_R} \int_{\mathscr{B}_R} \big(|u_0 - u'_\Delta| + |u_0 - u'|\big)\omega_\mu(0, t', x, x')\, dx\, d\pi'. \end{aligned}$$

According to properties (10)–(11) and (21)–(22) respectively for u and u_Δ, there exists a constant C such that the integrals in the right-hand side are bounded by $C_R C\mu$, which completes the proof of Lemma 4.3. ∎

Let us now collect relations (25) and (26) and consider the Kuznetsov inequality of Lemma 4.3. It follows, for any strictly positive real μ and for all t_n in $[\mu, T]$,

$$\begin{aligned} \|u(t_n, \cdot) - u_\Delta(t_n, \cdot)\|_{\mathscr{B}_{\overline{R}-\mathscr{N}t_n}} \leq\ & C\Big(\Big(\Delta t + \frac{\Delta t}{\mu} + \mu\Big) C_R + \mu \max(\mu, R)^{p-1}\Big) \\ &+ 2\sum_{m=0}^{n-1} \int_{\mathscr{K}^{m,\, m+1/2}_R} \int_{\mathscr{K}^n_R} \operatorname{sign}(u - u'_\Delta)(g(u) \\ &- g(u'_\Delta))\omega_\mu\, d\pi\, d\pi'. \end{aligned}$$

To bound the integral in the above inequality, we use first the fact that the function g is Lipschitzian on $\mathbb{R}$. Then, estimates (21)–(22) on the space and time variations for u_Δ lead to the existence of a constant C such that

$$\|u(t_n,\cdot)-u_\Delta(t_n,\cdot)\|_{\mathscr{B}_{\overline{R}-\mathcal{N}t_n}} \le C\Big(\Big(\Delta t+\frac{\Delta t}{\mu}+\mu\Big)C_R+\mu\max(\mu,R)^{p-1}\Big)$$
$$+C\int_0^{t_n}\|u(t,\cdot)-u_\Delta(t,\cdot)\|_{\mathscr{B}_{\overline{R}-\mathcal{N}t}}\,dt.$$

Just note that if t_n belongs to $[0,\mu[$, thanks to (11) and (22), we also have

$$\|u(t_n,\cdot)-u_\Delta(t_n,\cdot)\|_{\mathscr{B}_{\overline{R}-\mathcal{N}t_n}} \le C_R C\mu.$$

This together with the previous Gronwall-type inequality gives the desired result for $\mu=\sqrt{\Delta t}$, which concludes the proof of Theorem 2.1.

5. THE BOUNDARY-VALUE PROBLEM

5.1. *Mathematical Framework*

We now focus on the Dirichlet problem for the first-order quasi-linear operator $\mathbb{H}(\cdot)$ related to a forced unilateral constraint. Hence, let us introduce a bounded domain Ω of $\mathbb{R}^p$ with a Lipschitz boundary Γ. We denote $Q=]0,T[\times\Omega$ and $\Sigma=]0,T[\times\Gamma$. Then, given a real a and two measurable functions u_0, u_B such that $a\le u_0$ a.e. on Ω and $a\le u_B$ a.e. on Σ, we consider the free boundary problem: find u satisfying

$$a\le u \qquad \text{a.e. on } Q, \tag{27}$$

$$\mathbb{H}(u)\ge 0 \quad \text{and} \quad (u-a)\mathbb{H}(u)=0 \qquad \text{on } Q, \tag{28}$$

$$u=u_B \qquad \text{on an (unknown) part of } \Sigma, \tag{29}$$

$$u(0,\cdot)=u_0 \qquad \text{a.e. on } \Omega. \tag{30}$$

To apply the same time-splitting method as in the case of the Cauchy problem, we first have to make precise some fundamental properties for the solution of unilateral problem (27)–(30). They can be deduced from the results established in [12] for a positiveness constraint and a Dirichlet homogeneous boundary condition. To do so, we assume that hypothesis (h) of Section 1 is fulfilled. Moreover, we turn (h') into:

(h'') u_0 *(resp.* u_B*) belongs to* $L^\infty(\Omega)\cap BV(\Omega)$*(resp.* $L^\infty(\Sigma)$*), where, for any bounded subset* $\mathcal{O}$ *of* $\mathbb{R}^q$, $BV(\mathcal{O})$ *is the Banach space of integrable functions on* $\mathcal{O}$ *with bounded total variation on* $\mathcal{O}$*, used with the norm*

$$\|w\|_{BV(\mathcal{O})}=\|w\|_{L^1(\mathcal{O})}+TV_{\mathcal{O}}(w).$$

Under these conditions, by referring to the notation introduced in Section 1, we have:

THEOREM 5.1. *Obstacle problem* (27)–(30) *has a unique entropy solution* $u(t,x)$ *which belongs to* $L^\infty(Q) \cap \mathscr{C}([0,T]; L^1(\Omega)) \cap BV(Q)$, *satisfying* (27) *and* (30) *and characterized by the formulation*, $\forall\, k \in [a, +\infty[, \forall\, \phi \in \mathscr{C}^1_{c,+}(]0,T[\times\overline{\Omega})$,

$$\int_Q \mathscr{L}(u,k,\phi)\,dx\,dt - \int_\Sigma \operatorname{sign}(u_B - k)\big(\vec{f}(\gamma_u) - \vec{f}(k)\big)\cdot \vec{n}\phi\; d\mathrm{H}^p \geq 0, \quad (31)$$

where $\vec{n}$ *is the unit outward normal vector on* Σ, H^p *is the* p*-dimensional Hausdorff measure on* Σ, *and* γ *is the linear continuous mapping from* $BV(Q)$ *into* $L^1(\Sigma)$.

Let $\mathscr{I}(t)$ and $\mathscr{H}(t)$ be the two operators corresponding to the exact resolution of ODI (4) and homogeneous equation (3) linked to Dirichlet boundary condition (29). Thanks to the classical properties of $\mathscr{H}(t)$ (see, e.g., [1]) and those of $\mathscr{I}(t)$ (resulting from Theorem 3.2 and Proposition 3.1 by turning $\mathbb{R}^p$ and $\mathscr{B}_R$ into Ω since the initial data is an integrable function on the whole Ω) we are able to announce:

PROPOSITION 5.1. *The function* u_Δ *defined by* (18) *is an element of* $L^\infty(Q) \cap \mathscr{C}([0,T]; L^1(\Omega)) \cap BV(Q) \cap L^\infty(0,T; BV(\Omega))$. *Moreover there exists a constant* C, *such that*

$$\forall\, t \in [0,T], \qquad |u_\Delta(t,x)| \leq \widetilde{M}(t) \qquad \textit{for a.e. } x \textit{ of } \Omega, \quad (32)$$

$$\textit{for a.e. } t \textit{ in } [0,T], \qquad \|u_\Delta(t,\cdot)\|_{BV(\Omega)} \leq C, \quad (33)$$

$$\forall\, h \in]0,T[, \forall\, t \in [0, T-h], \quad \|u_\Delta(t+h,\cdot) - u_\Delta(t,\cdot)\|_{L^1(\Omega)} \leq Ch, \quad (34)$$

where $\widetilde{M}(t) = (\|u_0\|_{L^\infty(\Omega)} + \|u_B\|_{L^\infty(\Sigma)})\exp(K_g t)$.

5.2. *Convergence Result*

Our aim in this section is to specify the behavior, when Δt goes to 0^+ (or equally when N goes to $+\infty$), of the time-fractional step function $\mathscr{F}_s(t)u_0$ defined a.e. on Q through (18). In fact, only a convergence property is demonstrated. Indeed, we have not been able to develop the comparison method used Section 4.2—based on BV-properties. This reasoning can still be developed inside Ω. However, the properties of the trace operator γ from $BV(Q)$ into $L^1(\Sigma)$ do not suffice to extend the previously used techniques to boundary integrals. Thus, the next property holds:

THEOREM 5.2. *As* Δt *goes to* 0^+ *the sequence of time-fractional step functions* $(\mathscr{F}_s(t)u_0)_{\Delta t > 0}$ *converges a.e. on* Q, *strongly in* $L^q(Q)$, $1 \leq q < +\infty$, *and in* $\mathscr{C}([0,T]; L^1(\Omega))$ *to the entropy solution* u *of* (27)–(30).

Proof. First, on account of the compact embedding of $BV(\Omega)$ into $L^1(\Omega)$, the estimates of Proposition 5.1 and the Ascoli–Arzela theorem warrant the existence of a function $\hat{u}$ in $\mathscr{C}([0,T];L^1(\Omega))$ and a subsequence extracted from $(u_\Delta)_{\Delta t>0}$—still labelled $(u_\Delta)_{\Delta t>0}$—such that $(u_\Delta)_{\Delta t>0}$ tends to $\hat{u}$ in $\mathscr{C}([0,T];L^1(\Omega))$, when Δt goes to 0^+. Furthermore $\hat{u}$ is bounded on Q with bounded variations on Q and $(u_\Delta)_{\Delta t>0}$ strongly converges to $\hat{u}$ in $L^q(Q)$, $1 \le q < +\infty$. So to conclude, we have to check that $\hat{u}$ satisfies (31). That is why we need the next result, which can be obtained through work of Bardos *et al.* [1]:

LEMMA 5.1. *Let t_1 and t_2 be elements of* $[0,T]$, $t_1 < t_2$. *The weak entropy solution $w(t,x) = \mathscr{H}(t)u_0$ fulfills the regularized formulation* $\forall\, k \in \mathbb{R}$, $\forall\, \lambda > 0$, $\forall\, \phi \in \mathscr{C}^1_{c,+}(]0,T[\times\overline{\Omega})$,

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |w-k|_\lambda \partial_t\phi + \vec{G}_\lambda(w,k)\cdot\vec{\nabla}\phi\,dx\,dt \\ \ge \int_{t_1}^{t_2}\int_\Gamma\big\{\vec{G}_\lambda(u_B,k) - \operatorname{sign}_\lambda(u_B-k)\big(\vec{f}(u_B)-\vec{f}(\gamma_w)\big)\big\}\cdot\vec{n}\phi\,d\mathrm{H}^p \\ + \int_\Omega [|w-k|_\lambda\phi]_{t_1}^{t_2}\,dx,\end{aligned}$$

where the function $\operatorname{sign}_\lambda$ *is defined in the proof of Theorem* 3.1 *and*

$$|v|_\lambda = \int_0^v \operatorname{sign}_\lambda(\tau)\,d\tau$$

and

$$\forall\, i \in \{1,.,p\}, \qquad G_{\lambda,i}(v,k) = \int_k^v f_i'(\tau)\operatorname{sign}_\lambda(\tau-k)\,d\tau.$$

As a result, after the demonstration of Lemma 4.1, one can easily note that $\forall\, k \in [a,+\infty[$, $\forall\, \phi \in \mathscr{C}^1_{c,+}(]0,T[\times\overline{\Omega})$, and $\forall\, m \in \{0,1,\dots,N-1\}$,

$$\int_{t_m}^{t_{m+1/2}}\int_\Omega (|u_\Delta-k|_\lambda\partial_t\phi + 2\operatorname{sign}_\lambda(u_\Delta-k) \times g(u_\Delta)\phi)\,dx\,dt \ge \int_\Omega [|u_\Delta-k|_\lambda\phi]_{t_m}^{t_{m+1/2}}\,dx$$

and, according to Lemma 5.1,

$$\begin{aligned}\int_{t_{m+1/2}}^{t_{m+1}}\int_\Omega\left(\frac{1}{2}|u_\Delta-k|_\lambda\partial_t\phi + \vec{G}_\lambda(u_\Delta,k)\cdot\vec{\nabla}\phi\right)dx\,dt \\ \ge \int_{t_{m+1/2}}^{t_{m+1}}\int_\Gamma\big\{\vec{G}_\lambda(u_B,k) - \operatorname{sign}_\lambda(u_B-k)\big(\vec{f}(u_B)-\vec{f}(\gamma_{u_\Delta})\big)\big\}\cdot\vec{n}\phi\,d\mathrm{H}^p \\ + \frac{1}{2}\int_\Omega [|u_\Delta - k_\lambda\phi]_{t_{m+1/2}}^{t_{m+1}}\,dx.\end{aligned}$$

Following the idea of Crandall and Majda [5], the two previous relations are added. Then, the summation over m gives

$$\int_Q \left\{ \frac{1}{2}|u_\Delta - k|_\lambda \partial_t \phi + \chi_N \vec{G}_\lambda(u_\Delta, k) \cdot \vec{\nabla}\phi \right\} dx\, dt$$
$$+ \int_Q (1-\chi_N)\mathrm{sign}_\lambda(u_\Delta - k) g(u_\Delta)\, dx\, dt$$
$$\geq \int_\Sigma \chi_N \{ \vec{G}_\lambda(u_B, k) - \ \mathrm{sign}_\lambda(u_B - k)(\vec{f}(u_B) - \vec{f}(\gamma_{u_\Delta})) \} \cdot \vec{n}\phi\, d\mathrm{H}^p, \quad (35)$$

where χ_N denotes the characteristic function of the set

$$\bigcup_{m=0}^{N-1} \left\{ t \in [0, T],\ \left(m + \frac{1}{2}\right)\Delta t \leq t \leq (m+1)\Delta t \right\}.$$

The convergence properties of the sequence $(u_\Delta)_{\Delta t > 0}$, the fact that $(1 - \chi_N)_{N\in\mathbb{N}}$ and $(\chi_N)_{N\in\mathbb{N}}$ tend to $\frac{1}{2}$ weakly in $L^2(0, T)$ as N goes to $+\infty$ (see [5]) are used to take the Δt-limit in the left-hand side of the previous inequality. Let us now concentrate on the boundary integral

$$I = \int_\Sigma \chi_N \ \mathrm{sign}_\lambda(u_B - k)\vec{f}(\gamma_{u_\Delta}) \cdot \vec{n}\phi\, d\mathrm{H}^p.$$

Let w be an element of $H^1(Q) \cap L^\infty(Q)$ such that $w_{/\Gamma} = u_B$. For each value of the strictly positive real δ, we introduce ρ_δ, an approximation of the characteristic function of Ω—based on that introduced in [1]—such that

$$\rho_\delta \in \mathscr{C}^2(\overline{\Omega}),\ \ \rho_\delta = 0 \quad \text{on } \Gamma, \quad 0 \leq \rho_\delta \leq 1,$$
$$\rho_\delta = 1 \quad \text{on } \{x \in \Omega,\ \mathrm{dist}(x, \Gamma) \geq \delta\}, \qquad \|\vec{\nabla}\rho_\delta\|_\infty \leq \frac{C}{\delta}.$$

Therefore, Green's formula leads to

$$I = \int_Q \chi_N (1-\rho_\delta)\phi\, d\mu_\Delta + \int_Q \chi_N\ \mathrm{sign}_\lambda(w-k)\vec{f}(u_\Delta) \cdot \vec{\nabla}[(1-\rho_\delta)\phi]\, dx\, dt$$
$$= I_1 + I_2,$$

where $d\mu_\Delta$ is the positive Radon measure associated with the distribution $\vec{\nabla}[\mathrm{sign}_\lambda(w-k)\vec{f}(u_\Delta)]$, since u_Δ is an element of $L^\infty(Q) \cap BV(Q)$.

The real parameter δ still being fixed, the study of I_1 refers to the definition of u_Δ on each interval $[t_{m+1/2}, t_{m+1}]$, which provides a more convenient expression to pass to the limits in I_1. Thus, for any function Φ of $\mathscr{C}_c^1(Q)$,

$$\int_Q \chi_N \{ u_\Delta \partial_t \Phi + \vec{f}(u_\Delta) \cdot \vec{\nabla}\Phi \}\, dx\, dt = 0.$$

By density this equality is still fulfilled for any test-function Φ of $H_0^1(Q)$, so that it is possible to take Φ equal to $\rho_\epsilon \operatorname{sign}_\lambda(w-k)\zeta$, where ϵ is a positive parameter and ζ belongs to $\mathscr{C}^1_{c,+}(]0,T[\times\overline{\Omega})$. It follows that

$$\int_Q \chi_N \operatorname{sign}_\lambda(w-k)\big\{u_\Delta \rho_\epsilon \partial_t \zeta + \vec{f}(u_\Delta)\cdot\vec{\nabla}[\rho_\epsilon\zeta]\big\}\,dx\,dt$$
$$+\int_Q \chi_N\,\rho_\epsilon\zeta\big\{\partial_t \operatorname{sign}_\lambda(w-k) + \vec{f}(u_\Delta)\cdot\vec{\nabla}\operatorname{sign}_\lambda(w-k)\big\}\,dx\,dt = 0.$$

An integration by parts in the first term gives

$$\int_Q \chi_N \rho_\epsilon \zeta\,d\mu_\Delta = \int_Q \chi_N \operatorname{sign}_\lambda(w-k)u_\Delta\rho_\epsilon\partial_t\zeta\,dx\,dt$$
$$+\int_Q \chi_N\rho_\epsilon\zeta\big\{\partial_t \operatorname{sign}_\lambda(w-k)$$
$$+\vec{f}(u_\Delta)\cdot\vec{\nabla}\operatorname{sign}_\lambda(w-k)\big\}\,dx\,dt.$$

When ϵ goes to 0^+, ρ_ϵ converges to 1 everywhere on Ω. So, for any function ζ of $\mathscr{C}^1_{c,+}(]0,T[\times\overline{\Omega})$,

$$\int_Q \chi_N\zeta\,d\mu_\Delta = \int_Q \chi_N \operatorname{sign}_\lambda(w-k)u_\Delta\partial_t\zeta\,dx\,dt$$
$$+\int_Q \chi_N\zeta\big\{\partial_t \operatorname{sign}_\lambda(w-k) + \vec{f}(u_\Delta)\cdot\vec{\nabla}\operatorname{sign}_\lambda(w-k)\big\}\,dx\,dt.$$

In the latter relation one may choose ζ equal to $(1-\rho_\delta)\phi$. This provides a new expression for I_1 in which we are able to take the Δt-limit to obtain

$$\lim_{\Delta t\to 0^+} I_1 = \int_Q \frac{1}{2}\operatorname{sign}_\lambda(w-k)\hat{u}(1-\rho_\delta)\partial_t\phi dxdt$$
$$+\frac{1}{2}\int_Q \zeta(1-\rho_\delta)\phi\big\{\partial_t \operatorname{sign}_\lambda(w-k)$$
$$+\vec{f}(\hat{u})\cdot\vec{\nabla}\operatorname{sign}_\lambda(w-k)\big\}dxdt.$$

What is more, the real parameter δ still being fixed, it is possible to take the limit with respect to Δt in the integral I_2. Thus

$$\lim_{\Delta t\to 0^+} I_2 = \int_Q \frac{1}{2}\operatorname{sign}_\lambda(w-k)\vec{f}(\hat{u})\cdot\vec{\nabla}[(1-\rho_\delta)\phi]\,dx\,dt.$$

The Green formula is used to transform the previous expression into

$$\int_\Sigma \frac{1}{2}\operatorname{sign}_\lambda(u_B-k)\vec{f}(\gamma_{\hat{u}})\phi\,d\mathrm{H}^p - \int_Q \frac{1}{2}(1-\rho_\delta)\phi\,d\tilde{\mu},$$

where $d\tilde{\mu}$ is the positive Radon measure associated with the distribution $\vec{\nabla}[\operatorname{sign}_\lambda(w-k)\vec{f}(\hat{u})]$.

Finally, when δ goes to 0^+, $(1-\rho_\delta)$ converges to 0^+ simply on Ω. So,

$$\lim_{\delta\to 0^+}\lim_{\Delta t\to 0^+} I = \int_\Sigma \frac{1}{2}\operatorname{sign}_\lambda(u_B - k)\vec{f}(\gamma_{\hat{u}})\phi\, d\mathrm{h}^p.$$

By passing to the limit on N in (35) we get

$$\int_Q |\hat{u}-k|_\lambda \partial_t\phi + \vec{G}_\lambda(\hat{u},k)\cdot\vec{\nabla}\phi\, dx\, dt + \int_Q \operatorname{sign}_\lambda(\hat{u}-k)g(\hat{u})\, dx\, dt$$

$$\geq \int_\Sigma \{\vec{G}_\lambda(u_B,k) - \operatorname{sign}_\lambda(u_B-k)(\vec{f}(u_B)-\vec{f}(\gamma_{\hat{u}}))\}\cdot\vec{n}\phi\, d\mathrm{H}^p.$$

Entropy formulation (31) for $\hat{u}$ is obtained by taking the limit with respect to λ through the Lebesgue-dominated convergence theorem, which completes the proof of Theorem 5.2. ∎

6. CONCLUSION AND PROSPECTS

So, we have proved that the time-splitting method, used classically to compute discontinuous solutions of nonhomogeneous scalar conservation laws, can also be developed to approximate the solution of a unilateral obstacle problem for a first-order hyperbolic operator. The study of the error estimate for the theoretical splitting process provides a construction of numerical solutions to (1) and (2). Indeed in a one-space dimension, we have proposed in [13] a numerical method based on a truncation of the standard forward Euler scheme for the numerical computation of the solution to ODI (4). The continuous operator $\mathscr{H}$ is approximated by a monotone finite scheme consistent with (3). This framework allows one to refer to Tang and Teng's work [20] and provide an L^1-error bound between the computational and the exact solution to (1) and (2) which is in $\mathscr{O}(\sqrt{h})$, where h is the discretisation parameter. Further developements are now necessary to obtain the same bound in higher space dimensions.

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